POISSON QUEUEING SYSTEM

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POISSON QUEUEING SYSTEMS

The queueing system where the distribution of arrival and the departure both are assumed to be Poisson or the distribution of inter-arrival time and service time are assumed to be Exponentially distributed are called as the Poisson queuing system. The main Poisson queuing systems are:

1. $(M \mid M \mid 1):(\infty \mid FCFS)$
2. $(M \mid M \mid 1):(N \mid FCFS)$
3. $(M \mid M \mid C):(\infty \mid FCFS)$
4. $(M \mid M \mid C):(N \mid FCFS)$

DISTRIBUTION OF ARRIVALS

The models in which only arrivals are counted and no departure takes place are also called as Pure Birth Process. In queueing system the Pure Birth Process arises when an addition customer increase the arrival in the system.

The distribution of the arrival for Poisson queueing system is obtained under following axioms:

Axiom 1: The no. of arrivals in non-overlapping intervals of time are statistically independent, i.e., the process has independent increments.

Axiom 2: The probability that an arrival occurs within a very small time interval Δt is given by:

$$P_1[\Delta t] = \lambda \Delta t + o(\Delta t)$$

Axiom 3: The probability of more than one arrival during time interval Δt is negligibly small, i.e., $o(\Delta t)$

Therefore the probability of no arrival during Δt will be:

$$\Rightarrow P_0[\Delta t] = 1 - \lambda \Delta t - o(\Delta t)$$

Where $o(\Delta t)$ is a very small quantity, called as order of Δt, such that

$$\text{Lim}_{\Delta t \to 0} \{o(\Delta t)/ \Delta t\} = 0$$

DISTRIBUTION OF ARRIVAL

Let Pn[t] denotes the probability of n arrival during time interval of length t.

Now for n > 0:

$P_n[t + \Delta t]$ = Probability of n arrivals during time interval of length $(t + \Delta t)$

= P[n arrivals during time t, no arrival during time Δt]

+ P[(n–1) arrivals during time t, one arrival during time Δt]

+ P[(n–2)arrivals during time t, 2 arrival during time Δt]

+…+ P[no arrivals during time t, n arrival during time Δt]

$$=P_n[t].P_0[\Delta t] + P_{(n-1)}[t].P_1[\Delta t] + P_{(n-2)}[t].P_2[\Delta t] + \ldots + P_0[t].P_n[\Delta t]$$

$$= P_n[t].[1-\lambda\Delta t-o(\Delta t)] + P_{(n-1)}[t].[\lambda\Delta t+o(\Delta t)] + P_{(n-2)}[t].o[\Delta t] + \ldots + P_o(t).o(\Delta t)$$

On solving it, it is obtained that:

$$P_n[t+\Delta t] - P_n[t] = (\lambda\Delta t)\{P_{(n-1)}[t] - P_n[t]\} + o(\Delta t)$$

or,

$$\frac{P_n[t+\Delta t] - P_n[t]}{\Delta t} = \lambda\{P_{(n-1)}[t] - P_n(t)\} + \frac{o(\Delta t)}{\Delta t}$$

DISTRIBUTION OF ARRIVAL

Now, taking limit as $\Delta t \rightarrow 0$, it is obtained that;

$$\frac{dP_n[t]}{dt} = \lambda\{P_{(n-1)}[t] - P_n[t]\} \quad (1)$$

Now, consider the case when n=0,

$$P_0[t+\Delta t] = P[\text{there is no customer in the system at time } t+\Delta t]$$
$$= P[\text{no arrival during time } t].P[\text{no arrival during time } \Delta t]$$
$$= P_0[t].\{1 - \lambda\Delta t - o(\Delta t)\}$$

On solving it is obtained that;

$$P_0[t+\Delta t] - P_0[t] = \{-\lambda\Delta t - o(\Delta t)\}.P_0[t]$$

Or,

$$\frac{P_0[t+\Delta t] - P_0[t]}{\Delta t} = \left\{-\lambda - \frac{o(\Delta t)}{\Delta t}\right\}.P_0[t]$$

Taking limit as $\Delta t \rightarrow 0$, it is obtained that:

$$\frac{dP_0[t]}{dt} = -\lambda P_0[t] \quad (2)$$

DISTRIBUTION OF ARRIVAL

Equation (1) and (2) are called as differential-difference equations. Now multiplying the equation (1) by z^n (such that $|Z|\leq 1$), it is obtained that:

$$z^n \frac{dP_n[t]}{dt} = \lambda z^n\{P_{(n-1)}[t] - P_n[t]\}$$

Summing it over all n = 1, 2, …, ∞ and adding with equation (2), it is obtained that;

$$\frac{d}{dt}\sum_{n=0}^{\infty} z^n P_n[t] = \lambda z \sum_{n=0}^{\infty} z^n P_n[t] - \sum_{n=0}^{\infty} z^n P_n[t]$$

Or,

$$\frac{d}{dt}\sum_{n=0}^{\infty} z^n P_n[t] = \lambda z \sum_{n=0}^{\infty} z^n P_n[t] - \sum_{n=0}^{\infty} z^n P_n[t]$$

Also, it is known that the term $\sum_{n=0}^{\infty} z^n P_n[t]$ is called as probability generating function and denoted by *φ(z, t)*, therefore the equation would become:

$$d\varphi(z, t)/dt = \lambda(z - 1)\, \varphi(z, t)$$

Or, $\{1/\varphi(z, t)\}d\varphi(z, t)/dt = \lambda(z - 1)$

DISTRIBUTION OF ARRIVAL

On integrating with respect to t, it is obtained that:

$$\varphi(z, t) = Ce^{\{\lambda(z-1)\,t\}} \qquad (3)$$

C is the constant of the integration. It is know that there is no arrival at time $t=0$, therefore $P_n[0] = 0$ for all n >0 and $P_0[0] = 1$,

Therefore,

$$\varphi(z, 0) = P_0[0] + P_1[0]z + P_2[0]z^2 + \ldots + P_n[0]z^n + \ldots$$
$$= 1$$

Now, putting t=0 in equation (3),

$$\varphi(z, 0) = Ce^{\{\lambda(z-1)\,0\}}$$

Or, $1 = C$

Therefore,

$$\varphi(z, t) = e^{\{\lambda(z-1)\,t\}}$$

Now the different probabilities can be obtained by differentiating the probability generating function again and again and putting z = 0.

DISTRIBUTION OF ARRIVAL

$$\left.\frac{d\varphi(z,t)}{dt}\right|_{z=0} = P_1[t] => P_1[t] = \lambda t e^{-\lambda t}$$

$$\left.\frac{d^2\varphi(z,t)}{dt^2}\right|_{z=0} = 2!\,P_2[t] => P_2[t] = \frac{(\lambda t)^2}{2!}e^{-\lambda t}$$

$$\left.\frac{d^3\varphi(z,t)}{dt^3}\right|_{z=0} = 3!\,P_3[t] => P_3[t] = \frac{(\lambda t)^3}{3!}e^{-\lambda t}$$

By using mathematical induction,

$$\left.\frac{d^n\varphi(z,t)}{dt^n}\right|_{z=0} = n!\,P_n[t] => P_n[t] = \frac{(\lambda t)^n}{n!}e^{-\lambda t}$$

Hence, the distribution of the arrival is Poisson with parameter λt.

DISTRIBUTION OF INTER-ARRIVAL TIME

The inter-arrival time is the time gap between two successive arrivals. The inter-arrival time is also a random variable. Let T denotes the inter-arrival time, then:

$$\begin{aligned} P[T > t] &= P[\text{there is no arrival during time } t] \\ &= P_0[t] \\ &= e^{-\lambda t} \end{aligned}$$

If F(t) denoted the cumulative distribution function of T, then:

$$\begin{aligned} F(t) &= P[T \leq t] \\ &= 1 - P[T > t] \\ &= 1 - e^{-\lambda t} \end{aligned}$$

Which is the cumulative density function of exponential distribution with parameter λ.

Hence, If the distribution of arrival is Poisson, inter-arrival time will have exponential distribution with parameter λ.

DISTRIBUTION OF DEPARTURES

The models in which only departures are counted and no arrival takes place are also called as Pure Death Process. In queueing system the Pure Death Process arises when an addition customer is served the in the system.

The distribution of the departure for Poisson queueing system is obtained under following axioms:

Axiom 1: The no. of departures in non-overlapping intervals of time are statistically independent, i.e., the process has independent increments.

Axiom 2: The probability that an departure occurs within a very small time interval Δt is given by:

$$P_1[\Delta t] = \mu\Delta t + o(\Delta t)$$

Axiom 3: The probability of more than one departures during time interval Δt is negligibly small, i.e., $o(\Delta t)$.

Therefore the probability of no departure during Δt will be:

$$\Rightarrow P_0[\Delta t] = 1 - \mu\Delta t - o(\Delta t)$$

Where $o(\Delta t)$ is a very small quantity, called as order of Δt, such that

$$\text{Lim}_{\Delta t \to 0} \{o(\Delta t)/ \Delta t\} = 0$$

DISTRIBUTION OF DEPARTURES

Let $P_n[t]$ denotes the probability of n customers in the system at time t.

Now for n > 0:

$P_n[t + \Delta t]$ = Probability that n customers are in the system at time $(t + \Delta t)$

= P[n customers at time t, no departure during time Δt]

+ P[(n+1) customers at time t, one departure during time Δt]

+ P[(n+2) customers at time t, 2 departures during time Δt] +…

$$=P_n[t].P_0[\Delta t] + P_{(n+1)}[t].P_1[\Delta t] + P_{(n+2)}[t].P_2[\Delta t] + \ldots$$

$$= P_n[t].[1-\mu\Delta t-o(\Delta t)] + P_{(n+1)}[t].[\mu\Delta t+o(\Delta t)] + P_{(n+2)}[t].o[\Delta t] +\ldots$$

On solving it, it is obtained that:

$$P_n[t+ \Delta t] - P_n[t] = (\mu\Delta t)\{P_{(n-1)}[t] - P_n[t]\} + o(\Delta t)$$

or,

$$\frac{P_n[t+\Delta t]-P_n[t]}{\Delta t} = \mu\{P_{(n+1)}[t] - P_n[t]\} + \frac{o(\Delta t)}{\Delta t}$$

DISTRIBUTION OF DEPARTURES

Now, taking limit as Δt→0, it is obtained that;

$$\frac{dP_n(t)}{dt} = \mu\{P_{(n+1)}[t] - P_n[t]\} \qquad (1)$$

Now, consider the case when n=0,

$P_0[t+\Delta t] = P[\text{there is no customer in the system at time } t+\Delta t]$

$= P[\text{no customer during time } t].P[\text{no departure during time } \Delta t]$

$+ P[1 \text{ customer during time } t].P[1 \text{ departure during time } \Delta t] + \ldots$

$= P_0[t].1 + P_1[t].\{1 - \mu\Delta t - o(\Delta t)\} + o(\Delta t)$

On solving it is obtained that;

$$P_0[t+\Delta t] - P_0[t] = \{-\mu\Delta t - o(\Delta t)\}.P_1[t] + o(\Delta t)$$

Or,

$$\frac{P_0[t+\Delta t] - P_0[t]}{\Delta t} = -\mu P_1[t] + \frac{o(\Delta t)}{\Delta t}$$

Taking limit as $\Delta t \to 0$, it is obtained that:

$$\frac{dP_0[t]}{dt} = -\mu P_1[t] \qquad (2)$$

DISTRIBUTION OF DEPARTURES

Now for n=N,

$$P_N[t+\Delta t] = P[N \text{ customers at time } t \text{ and no departures during } \Delta t]$$

$$= P_N[t].[1 - \mu\Delta t - o(\Delta t)]$$

=> $$\frac{P_N[t+\Delta t] - P_N[t]}{\Delta t} = -\mu P_N[t] + \frac{o(\Delta t)}{\Delta t}$$

Taking limit Δt → 0, it will be obtained that:

$$\frac{dP_N[t]}{dt} = -\mu P_N[t] \qquad (3)$$

On solving the equation (1), (2) and (3) under the initial condition:

$$P_n[0] = \begin{cases} 1; & n = N \neq 0 \\ 0; & n \neq N \end{cases}$$

it will be obtained that:

$$P_n[t] = \frac{(\mu t)^{(N-n)} e^{-\mu t}}{(N-n)!}$$

Which is the truncated Poisson distribution.

DISTRIBUTION OF SERVICE TIME

The service time is the time required to serve a customer. The service time is also a random variable. Let T denotes the service time, then:

$$P[T > t] = P[\text{there is no departure during time } t]$$
$$= P[\text{There are N customers at time t in the system}]$$
$$= P_N[t]$$
$$= e^{-\mu t}$$

If F(t) denoted the cumulative distribution function of service time T, then:

$$P[T \leq t] = 1 - P[T > t]$$
$$= 1 - e^{-\mu t}$$

Which is the cumulative density function of exponential distribution with parameter μ.

Hence, If the distribution of departure is Poisson, Service time will have exponential distribution with parameter μ.

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Note: The derivations and examples consider in the present notes is taken from the reference book mentioned at serial no. 3.