DUALITY IN LPP

By,
Dr. Akash Asthana,
Assistant Professor,
University of Lucknow,
Lucknow

DUALITY IN LPP

Associated with every Linear Programming Problem there always exist another Linear Programming Problem which is based upon same data and having the same solution. This property of Linear Programming Problem is termed as **Duality in Linear Programming Problem**.

The original problem is called as **Primal Problem** and associated problem is called as **Dual problem**.

Any of these Linear Programming Problem can be taken as primal and other as dual therefore these problems are simultaneously called as **Primal-Dual Pair**.

FORMULATION OF DUAL PROBLEM

For the formulation of dual problem from the primal problem following steps are used:

1. Convert the constraints of given LPP in the standard form using slack and surplus variables only.
2. Identify the decision variables for the dual problem (Dual variables). The no. of dual variables will be equal to the no. of constraints in primal problem.
3. Write the objective function for the dual problem by taking the constants on the right hand side of primal constraints as the cost coefficients for the dual problem. If primal problem is maximization type then dual will be minimization type and vice-versa.
4. Define the constraints for the dual problem. The column constraint coefficients of primal problem will become the row constraint coefficients of dual problem. The cost coefficient of primal problem will be taken as the constants on the right hand side of dual constrains. If primal is of maximization type and dual constrains must be of '≥' type and If primal is of minimization type and dual constrains must be of '≤'type.
5. Dual variables will be unrestricted.

FORMULATION OF DUAL PROBLEM

Let's consider the Linear Programming Problem as a maximization problem as follows:

$$Max.\ Z = c_1x_1 + c_2x_2 + \dots + c_nx_n;$$

Subject to,

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le b_m$$
$$x_1, x_2, \dots, x_n \ge 0$$

Then its dual will be,

$$Min.\ Z_1 = b_1w_1 + b_2w_2 + \dots + b_mw_m;$$

Subject to,

$$a_{11}w_1 + a_{21}w_2 + \dots + a_{m1}w_m \ge c_1$$
$$a_{12}w_1 + a_{22}w_2 + \dots + a_{m2}w_m \ge c_2$$
$$\vdots$$
$$a_{1n}w_1 + a_{2n}w_2 + \dots + a_{mn}w_m \ge c_n$$
$$w_1, w_2, \dots, w_m \text{ will be unrestricted}$$

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FORMULATION OF DUAL PROBLEM

However it is better to use the matrix form of Linear Programming Problem for the formulation of dual problem. In matrix form the primal-dual pair can be written as:

Primal Problem	Dual Problem
Max $Z = C'X$ $AX = b$ $X \geq 0$	Min $Z_1 = b'W$ $A'W \geq c$ W unrestricted
Min $Z = C'X$ $AX = b$ $X \geq 0$	Max $Z_1 = b'W$ $A'W \leq c$ W unrestricted

EXAMPLE

Obtain the dual of the following Linear Programming Problem:

$$\text{Min. } Z = 2x_1 + 3x_2 + 4x_3$$

Subject to,

$$2x_1 + 3x_2 + 5x_3 \geq 2$$
$$3x_1 + x_2 + 7x_3 = 3$$
$$x_1 + 4x_2 + 6x_3 \leq 5$$
$$x_1, x_2 \geq 0, x_3 \text{ is unrestricted}$$

Now, this problem can be written in standard form as:

$$\text{Min. } Z = 2x_1 + 3x_2 + 4x'_3 - 4x''_3$$

Subject to,

$$2x_1 + 3x_2 + 5x'_3 - 5x''_3 - s_1 = 2$$
$$3x_1 + x_2 + 7x'_3 - 7x''_3 = 3$$
$$x_1 + 4x_2 + 6x'3 - 6x''_3 + s_2 = 5$$
$$x_1, x_2, x'_3, x''_3 \geq 0,$$

EXAMPLE

The given problem can be written in matrix form as:

$$Min.\, z = \begin{bmatrix} 2 & 3 & 4 & -4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x'_3 \\ x''_3 \\ s_1 \\ s_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 5 & -5 & -1 & 0 \\ 3 & 1 & 7 & -7 & 0 & 0 \\ 1 & 4 & 6 & -6 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x'_3 \\ x''_3 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\underline{X} \geq 0$$

EXAMPLE

As the number of constraints in the primal problem is 3, the number of dual variables would be 3. Let the dual variables are w_1, w_2 and w_3.

Now as the objective function of the primal problem is of minimization type the objective function of dual will be of maximization type. The objective function of dual problem would be:

$$Max.\, Z_1 = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

As the objective function of the primal problem is of minimization type the constraints of dual will contain '≤' sign. The constraints would be:

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 4 \\ 5 & 7 & 6 \\ -5 & -7 & -6 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 3 \\ 4 \\ -4 \\ 0 \\ 0 \end{bmatrix}$$

Dual variables would be unrestricted.

EXAMPLE

On multiplying the matrices we get:

$$Max.\ Z_1 = 2w_1 + 3w_2 + 4w_3$$

Subject to,

$$2w_1 + 3w_2 + w_3 \leq 2 \quad (1), \qquad 3w_1 + w_2 + 4w_3 \leq 3 \quad (2)$$

$$5w_1 + 7w_2 + 6w_3 \leq 4 \quad (3), \qquad -5w_1 - 7w_2 - 6w_3 \leq -4 \quad (4)$$

$$-w_1 \leq 0 \ (5) \qquad w_3 \leq 0 \quad (6)$$

Constraint (4) can be written as:

$$5w_1 + 7w_2 + 6w_3 \geq 4 \qquad (4)$$

Constraints (3) and (4) are contradictory, these will be true if and only if the equality holds, therefore constraints (3) and (4) would be combined into one constraint as:

$$5w_1 + 7w_2 + 6w_3 = 4$$

Constraint (5) would give $w_1 \geq 0$.

EXAMPLE

Therefore the dual problem would be:

$$Max.\ Z_1 = 2w_1 + 3w_2 + 4w_3$$

Subject to,

$$2w_1 + 3w_2 + w_3 \leq 2,$$

$$3w_1 + w_2 + 4w_3 \leq 3$$

$$5w_1 + 7w_2 + 6w_3 = 4$$

$$w_1 \geq 0,\ w_3 \leq 0,\ w_2 \text{ would be unrestricted}$$

THEOREM

Statement: Dual of Dual is Primal.

Proof: Without loss of generality we can take the primal problem as:

$$Max.\ Z = C'X$$

$$AX = b,$$

$$X \geq 0$$

Then dual of this problem would be:

$$Min.\ Z_1 = b'W$$

$$A'W \geq 0$$

$$W \text{ is unrestricted.}$$

Now the dual problem would be converted into standard form.

As W is unrestricted it can be expressed as the difference of two non-negatively restricted variables W_1 and W_2 such that:

$$W = W_1 - W_2,\ W_1 \geq 0 \text{ and } W_2 \geq 0$$

THEOREM (CONTD…)

Now the dual problem would become:

$$Min.\ Z_1 = b'(W_1 - W_2)$$

$$A'(W_1 - W_2) - Is = C$$

$$W_1 \geq 0,\ W_2 \geq 0,\ s \geq 0$$

It can also be written as:

$$Min. Z_1 = \begin{bmatrix} b' & -b' & 0 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ s \end{bmatrix}$$

$$\begin{bmatrix} A' & -A' & I \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ s \end{bmatrix} = C$$

$$\begin{bmatrix} W_1 \\ W_2 \\ s \end{bmatrix} \geq 0$$

THEOREM (CONTD…)

Now, the Dual of Dual would be:

$$\text{Max. } Z = C'Y$$

$$\begin{bmatrix} A \\ -A \\ I \end{bmatrix} Y \le \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$$

Y is unrestricted.

Or, the dual of dual can be written as:

$$Max.\ Z = C'Y$$

$$AY \le b$$

$$-AY \le -b \Rightarrow AY \ge b$$

$$Y \ge 0$$

First two constraints contradict each other, they will be true if and only if AY = b

As Y is just a notation for the dual variable it can be replaced by X, and the Dual of Dual would be:

$$Max.\ Z = C'X$$

$$AX = b$$

$$X \ge 0$$

Hence Dual of Dual is Primal.

RELATION BETWEEN THE SOLUTION OF PRIMAL AND DUAL PROBLEMS

Solution of Primal Problem	Solution of Dual Problem
Optimum	Optimum
Infeasible	Unbounded
Unbounded	Infeasible

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Note: The Methodology and examples consider in the present notes are taken from the reference book mentioned at serial no. 3.